

MEROMORPHIC MULTIVALENT CLOSE-TO-CONVEX FUNCTIONS

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1. Introduction. Recently the author [5] has discussed some properties of the class of p -valent regular close-to-convex functions, called $\mathcal{K}(p)$. It is the purpose of this paper to generalize some of these results to the meromorphic case.

Let $f(z)$ be meromorphic for $|z| < 1$ with q ($1 \leq q \leq p$) poles at the origin and $f(z) \neq 0$ for $|z| < 1$. We shall say that $f(z)$ is in $S_1^*(p)$ if there exists a ρ ($0 < \rho < 1$) such that for $z = re^{i\theta}$ ($\rho < r < 1$),

$$(1.1) \quad \operatorname{Re} \left[\frac{zf'(z)}{f(z)} \right] < 0$$

and

$$(1.2) \quad \int_0^{2\pi} d \arg f(z) = \int_0^{2\pi} \operatorname{Re} \left[\frac{zf'(z)}{f(z)} \right] d\theta = -2p\pi.$$

We shall say that $f(z)$ is in $S_2^*(p)$ if it is regular on $|z| = 1$ and if (1.1) and (1.2) hold for $|z| = 1$. If $f(z)$ is in $S_1^*(p)$, there exists a δ ($0 < \delta < 1$) such that $f(rz)$ is in $S_2^*(p)$ if $\delta < r < 1$.

We set $S^*(p) = S_1^*(p) \cup S_2^*(p)$ and say that a function in $S^*(p)$ is starlike of order p .

Condition (1.2) along with the argument principle implies that a function in $S^*(p)$ has exactly p poles in $|z| < 1$. It is easily seen that a function $f(z)$, meromorphic in $|z| < 1$, is in $S^*(p)$ if and only if the function $[f(z)]^{-1}$ is regular and p -valently starlike in $|z| < 1$. Since the reciprocal of a p -valent function is p -valent, a function in $S^*(p)$ is p -valent in $|z| < 1$. Also, using the fact that a regular p -valent starlike function can be written as the p th power of a regular univalent starlike function, it is easily seen that a function in $S^*(p)$ with p poles at the origin can be written as the p th power of a meromorphic univalent starlike function.

Let $F(z)$ be meromorphic in $|z| < 1$ with q ($1 \leq q \leq p$) poles at the origin and with at most p poles in $|z| < 1$. We shall say that $F(z)$ is in $\mathcal{K}_1^*(p)$ if there exists a function in $S^*(p)$ and a ρ ($0 < \rho < 1$) such that for $z = re^{i\theta}$ ($\rho < r < 1$)

$$(1.3) \quad \operatorname{Re} \left[\frac{zF'(z)}{f(z)} \right] > 0.$$

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We shall say that $F(z)$ is in $\mathcal{K}_2^*(p)$ if $F(z)$ is regular on $|z| = 1$ and if there exists a function $f(z)$ in $S_2^*(p)$ such that (1.3) is satisfied for $|z| = 1$. If $F(z)$ is in $\mathcal{K}_1^*(p)$, there exists a δ ($0 < \delta < 1$) such that $F(rz)$ is in $\mathcal{K}_2^*(p)$ if $\delta < r < 1$.

We set $\mathcal{K}^*(p) = \mathcal{K}_1^*(p) \cup \mathcal{K}_2^*(p)$ and say that a function in $\mathcal{K}^*(p)$ is close-to-convex of order p .

The class $\mathcal{K}^*(1)$ was defined by Libera and Robertson [4] and Pommerenke [7]. It was shown in both papers that a function in $\mathcal{K}^*(1)$ need not be univalent. To show that a function in $\mathcal{K}^*(p)$ need not be p -valent, let $F(z)$ be such that

$$\frac{zF'(z)}{z^p} = \frac{1+z^{2p}}{1-z^{2p}} \quad (|z| < 1).$$

Then

$$F(z) = -\frac{1}{pz^p} + \frac{2}{p}z^{2p} + \frac{2}{3p}z^{3p} + \dots \quad (0 < |z| < 1).$$

If $F(z)$ was p -valent, then

$$F(z^{1/p}) = -\frac{1}{pz} + \frac{2}{p}z + \dots \quad (0 < |z| < 1)$$

would be univalent, and so would

$$-pF(z^{1/p}) = \frac{1}{z} - 2z + \dots \quad (0 < |z| < 1).$$

But this is impossible, since the coefficient of z has modulus greater than 1. Thus $F(z)$ is at least $2p$ -valent.

Necessary and sufficient conditions for a function to be in $\mathcal{K}^*(1)$ have been given in [4] and [7]. In §2 we obtain necessary conditions for a function $F(z)$ to be in $\mathcal{K}^*(p)$ and show that these conditions with the added assumptions of regularity on $|z| = 1$ and $F'(z) \neq 0$ in $|z| \leq 1$ are sufficient.

Recently, Royster [8] has shown that if

$$f(z) = \sum_{n=-p}^{\infty} a_n z^n \quad (0 < |z| < 1)$$

is in $S^*(p)$ then $|a_n| = O(1/n)$. In §3 we will extend this result to functions in $\mathcal{K}^*(p)$ with p poles at the origin. This result was obtained for $\mathcal{K}^*(1)$ by Libera and Robertson [4] and Pommerenke [7].

2. The class $\mathcal{K}^*(p)$.

THEOREM 1. *If $F(z)$ is in $\mathcal{K}^*(p)$, then there exists ρ ($0 < \rho < 1$) such that for $z = re^{i\theta}$ ($\rho < r < 1$)*

$$(2.1) \quad \int_0^{2\pi} d \arg d F(z) = \int_0^{2\pi} \frac{d}{d\theta} \arg \left[r e^{i\theta} F'(r e^{i\theta}) \right] d\theta = -2p\pi$$

and for any θ_1 and θ_2 with $0 \leq \theta_1 < \theta_2 \leq 2\pi$

$$(2.2) \quad \int_{\theta_1}^{\theta_2} d \arg d F(z) < \pi.$$

Proof. Suppose $F(z)$ is in $\mathcal{K}_1^*(p)$. Then there exists $f(z)$ in $S^*(p)$ and ρ ($0 < \rho < 1$) such that (1.1), (1.2) and (1.3) hold for $\rho < |z| < 1$.

Since $\operatorname{Re} [zF'(z)/f(z)] > 0$ for $|z| = r$ ($\rho < r < 1$), we may define

$$\arg [zF'(z)/f(z)]$$

to be single valued and continuous for $|z| = r$ and such that

$$(2.3) \quad \left| \arg \frac{zF'(z)}{f(z)} \right| < \frac{\pi}{2} \quad (|z| = r).$$

Furthermore, since $zF'(z) \neq 0$ for $|z| = r$, we may define $\arg [zF'(z)]$ to be single valued and continuous for $|z| = r$. Since $f(z) = [f(z)/zF'(z)] [zF'(z)]$, we may define $\arg [f(z)] = \arg [zF'(z)] - \arg [zF'(z)/f(z)]$ to be a single valued and continuous determination of $\arg [f(z)]$ for $|z| = r$. Then

$$(2.4) \quad \left| \arg zF'(z) - \arg f(z) \right| = \left| \arg \frac{zF'(z)}{f(z)} \right| < \frac{\pi}{2} \quad (|z| = r).$$

It is easily seen that (2.4) implies

$$(2.5) \quad -\pi + \int_{\theta_1}^{\theta_2} d \arg f(z) < \int_{\theta_1}^{\theta_2} d \arg d F(z) < \pi + \int_{\theta_1}^{\theta_2} d \arg f(z)$$

for $\theta_1 < \theta_2$ and $|z| = r$. Since $f(z)$ is in $S^*(p)$,

$$\int_{\theta_1}^{\theta_2} d \arg f(z) < 0 \quad (|z| = r).$$

Thus we obtain (2.2) for $|z| = r$ from the right side of (2.5). Letting $\theta_1 = 0$ and $\theta_2 = 2\pi$ in (2.5) and noting that

$$\int_0^{2\pi} d \arg f(z) = -2p\pi$$

we obtain

$$(2.6) \quad -(2p+1)\pi < \int_0^{2\pi} d \arg d F(z) < -(2p-1)\pi.$$

However, the integral appearing in (2.6) is an integral multiple of 2π . Thus (2.1) holds for $|z| = r$. Since r was arbitrary ($\rho < r < 1$) (2.1) and (2.2) hold for $\rho < |z| < 1$.

If $F(z)$ is in $\mathcal{K}_2^*(p)$, then the preceding argument with $r = 1$ shows that (2.1) and (2.2) hold for $|z| = 1$. But since $F(z)$ is regular near $|z| = 1$, we can show the existence of a ρ ($0 < \rho < 1$) such that (2.1) and (2.2) hold for $\rho < |z| \leq 1$.

Using (2.1) and the argument principle we immediately obtain the following corollary.

COROLLARY 1. *If $F(z)$ is in $\mathcal{K}^*(p)$, then $F'(z)$ has at least $(p + 1)$ poles in $|z| < 1$ and if $F'(z) \neq 0$ for $|z| < 1$, then $F'(z)$ has exactly $(p + 1)$ poles in $|z| < 1$.*

THEOREM 2. *Let $F(z)$ be meromorphic in $|z| < 1$ with q ($1 \leq q \leq p$) poles at the origin. If $F'(z) \neq 0$ for $0 < |z| \leq 1$ and $F(z)$ is regular on $|z| = 1$ and if (2.1) and (2.2) hold for $|z| = 1$, then $F(z)$ is in $\mathcal{K}_2^*(p)$.*

Proof. Consider the function $G(z)$, regular for $|z| = 1$, given by

$$G(z) = \int_0^z \frac{dz}{z^2 F'(z)} = b_q z^q + \dots.$$

Since $zF'(z) \neq 0$ for $|z| = 1$ we may define $\arg [zF'(z)]$ to be single valued and continuous for $|z| = 1$. Since $zG'(z) = [zF'(z)]^{-1}$, we may define $\arg zG'(z) = -\arg zF'(z)$. Thus, for $|z| = 1$

$$\int_0^{2\pi} d \arg d G(z) = 2p\pi$$

and

$$\int_{\theta_1}^{\theta_2} d \arg d G(z) > -\pi \quad (\theta_1 < \theta_2).$$

The author has shown (Theorem 3 [5]) that under these conditions $G(z)$ is in $\mathcal{K}(p)$. That is, there exists $g(z)$, regular for $|z| \leq 1$ such that

$$\operatorname{Re} \left[\frac{zg'(z)}{g(z)} \right] > 0 \quad (|z| = 1)$$

and

$$\operatorname{Re} \left[\frac{zG'(z)}{g(z)} \right] > 0 \quad (|z| = 1).$$

The function $f(z) = [g(z)]^{-1}$ is in $S^*(p)$ and

$$\frac{zG'(z)}{g(z)} = zG'(z)f(z) = \frac{f(z)}{zF'(z)}.$$

Thus

$$\operatorname{Re} \left[\frac{zF'(z)}{f(z)} \right] = \operatorname{Re} \left[\frac{g(z)}{zG'(z)} \right] > 0 \quad (|z| = 1).$$

Therefore $F(z)$ is in $\mathcal{K}_2^*(p)$.

Using the same procedure as above and by appealing to Theorem 2 [5], we may remove the condition of regularity on $|z| = 1$, if $q = p$. We thus have the following theorem.

THEOREM 3. *Let*

$$F(z) = \sum_{n=-p}^{\infty} a_n z^n \quad (0 < |z| < 1)$$

be meromorphic for $|z| < 1$ and $F'(z) \neq 0$. If there exists a ρ ($0 < \rho < 1$) such that (2.1) and (2.2) hold for $\rho < |z| < 1$, then $F(z)$ is in $\mathcal{K}^(p)$.*

We will have need of the next lemma in what follows.

LEMMA 1. *Let $F(z)$ be in $\mathcal{K}_2^*(p)$. Then, there exists a function*

$$f(z) = \sum_{n=-p}^{\infty} b_n z^n \quad (0 < |z| < 1) \quad (|b_{-p}| = 1),$$

in $S_2^(p)$, such that*

$$\operatorname{Re} \left[\frac{zF'(z)}{f(z)} \right] > 0 \quad (|z| = 1).$$

Proof. There exists a function $g(z)$ in $S_2^*(p)$ with s poles ($1 \leq s \leq p$) at the origin such that,

$$\operatorname{Re} \left[\frac{zF'(z)}{g(z)} \right] > 0 \quad (|z| = 1).$$

The function $g(z)$ has $(p-s)$ nonzero poles in $|z| < 1$. Let $\alpha_1, \alpha_2, \dots, \alpha_{p-s}$ be these poles and let

$$h(z) = z^{s-p} \prod_{i=1}^{p-s} (z - \alpha_i) (1 - \bar{\alpha}_i z)$$

and

$$f(z) = h(z)g(z) = \sum_{n=-p}^{\infty} c_n z^n \quad (0 < |z| < 1).$$

Since $[zh'(z)/h(z)]$ is purely imaginary on $|z| = 1$ and $\operatorname{Re} [zg'(z)/g(z)] < 0$ for $|z| = 1$, then $\operatorname{Re} [zf'(z)/f(z)] < 0$ for $|z| = 1$. Furthermore, since $f(z)$ has p poles in $|z| \leq 1$, all of them at the origin, and since $f(z) \neq 0$ in $|z| \leq 1$,

$$\int_0^{2\pi} \operatorname{Re} \left[\frac{e^{i\theta} f'(e^{i\theta})}{f(e^{i\theta})} \right] d\theta = -2p\pi.$$

Thus, $f(z)$ in $S_2^*(p)$. Furthermore,

$$\frac{zF'(z)}{f(z)} = \frac{z^{p-s}zF'(z)}{\prod_{i=1}^{p-s}(z-\alpha_i)(1-\bar{\alpha}_iz)g(z)}.$$

But $z^{p-s}[\prod_{i=1}^{p-s}(z-\alpha_i)(1-\bar{\alpha}_iz)]^{-1}$ is real and positive on $|z|=1$. Therefore

$$\operatorname{Re} \left[\frac{zF'(z)}{f(z)} \right] > 0 \quad (|z|=1).$$

Replacing $f(z)$ by $1/|c_{-p}|f(z)$, the proof of the lemma is completed.

THEOREM 4. *If $F(z)$ is in $\mathcal{K}^*(p)$ and has all its poles at the origin, then necessarily it has p poles there and $F'(z) \neq 0$ for $|z| < 1$.*

Proof. Suppose

$$F(z) = \sum_{n=-q}^{\infty} a_n z^n \quad (0 < |z| < 1) \quad (1 \leq q \leq p).$$

There exists a ρ ($0 < \rho < 1$) such that $F(rz)$ is in $\mathcal{K}_2^*(p)$ if $\rho < r < 1$. By Lemma 1, there exists

$$f(z) = \sum_{n=-p}^{\infty} b_n z^n \quad (0 < |z| < 1)$$

in $S_2^*(p)$ such that

$$\operatorname{Re} \left[\frac{rzF'(rz)}{f(z)} \right] > 0 \quad (|z|=1).$$

Since $f(z) \neq 0$ for $|z| \leq 1$,

$$\frac{rzF'(rz)}{f(z)} = \sum_{n=p-q}^{\infty} c_n z^n$$

is regular for $|z| \leq 1$. Thus,

$$\operatorname{Re} \left[\frac{rzF'(rz)}{f(z)} \right] > 0 \quad (|z| \leq 1).$$

Therefore, we must necessarily have $q = p$ and $F'(rz) \neq 0$ for $|z| \leq 1$. Thus, $F'(z) \neq 0$ for $|z| \leq r$. Since r was arbitrary ($\rho < r < 1$), $F'(z) \neq 0$ for $|z| < 1$.

If $F(z)$ has all its poles at the origin we may improve Lemma 1 by removing the condition of regularity on $|z|=1$.

LEMMA 2. *Let*

$$F(z) = \sum_{n=-p}^{\infty} a_n z^n \quad (0 < |z| < 1)$$

be in $\mathcal{K}^*(p)$. Then there exists

$$f(z) = \sum_{n=-p}^{\infty} b_n z^n \quad (0 < |z| < 1) \quad (|b_{-p}| = 1)$$

in $S^*(p)$ such that

$$\operatorname{Re} \frac{zF'(z)}{f(z)} > 0 \quad (|z| < 1).$$

Proof. There exists a ρ ($0 < \rho < 1$) such that the function $F_\rho(z) = F(\rho z)$ is in $\mathcal{K}_2^*(p)$ if $\rho < r < 1$. Then by Lemma 1 there exists

$$f_r(z) = \sum_{n=-p}^{\infty} c_n z^n \quad (0 < |z| < 1) \quad (|c_{-p}| = 1)$$

in $S_2^*(p)$, such that

$$\operatorname{Re} \left[\frac{zF'_r(z)}{f_r(z)} \right] > 0 \quad (|z| \leq 1).$$

Let r_i ($\rho < r_i < 1$) be an increasing sequence tending to 1. The functions $[f_{r_i}(z)]^{-1}$ are regular and p -valently starlike and have the moduli of their first p coefficients fixed. The class of regular and p -valently starlike functions with the moduli of their first p coefficients fixed forms a normal family of functions [1]. Thus, we can obtain a subsequence $[f_{r_{i_k}}(z)]^{-1}$ tending uniformly in every closed subset of $|z| < 1$ to a function $f(z)$ regular and p -valently starlike and such that

$$f(z) = \sum_{n=p}^{\infty} d_n z^n \quad (|z| < 1) \quad (|d_p| = 1).$$

Since $F_{r_{i_k}}(z)$ tends to $F(z)$ as r_{i_k} tends to 1 and since

$$\operatorname{Re}[zF_{r_{i_k}}(z)[f_{r_{i_k}}(z)]^{-1}] > 0 \quad \text{for } |z| < 1$$

we have

$$\operatorname{Re} [zF'(z)f(z)] > 0 \quad \text{for } |z| < 1.$$

But

$$g(z) = [f(z)]^{-1} = \sum_{n=-p}^{\infty} b_n z^n \quad (0 < |z| < 1) \quad (|b_{-p}| = 1)$$

is in $S^*(p)$ and

$$\operatorname{Re} \left[\frac{zF'(z)}{g(z)} \right] = \operatorname{Re} [zF'(z)f(z)] > 0 \quad \text{for } |z| < 1.$$

3. The coefficients of a function in $\mathcal{K}^*(p)$. We will make use of the following lemma, proven by Royster [8] and the author [6].

LEMMA 3. *Let*

$$f(z) = \sum_{n=-p}^{\infty} b_n z^n \quad (0 < |z| < 1) \quad (|b_{-p}| = 1)$$

be in $S^*(p)$, then for $n \geq 1$

$$|b_n| \leq \frac{2p}{(n+p)\sqrt{p}} \left(\sum_{k=-p}^{-1} |k| |b_k|^2 \right)^{1/2}.$$

The following lemma was proven for $p = 1$ by Pommerenke [7].

LEMMA 4. *Let*

$$F(z) = \frac{1}{z^p} + \sum_{n=-(p-1)}^{\infty} a_n z^n \quad \text{and} \quad f(z) = \frac{e^{i\beta}}{z^p} + \sum_{n=-(p-1)}^{\infty} b_n z^n, \quad (0 < |z| < 1)$$

and let $U(z) = \operatorname{Re} [zF'(z)/f(z)]$, then for $r < 1$

$$(3.1) \quad \begin{aligned} na_n = & -pe^{-i\beta}b_n + \frac{1}{\pi} \int_0^{2\pi} \left[U(re^{i\theta}) \frac{e^{-in\theta}}{r^n} \right] \\ & \times \left[f(re^{i\theta}) - \sum_{k=n}^{\infty} b_k (re^{i\theta})^k \right] d\theta. \end{aligned}$$

Proof. Let

$$\frac{zF'(z)}{f(z)} = -pe^{-i\beta} + \sum_{k=1}^{\infty} C_k z^k \quad (|z| < 1).$$

Then

$$\begin{aligned} \frac{-p}{z^p} + \sum_{n=-(p-1)}^{\infty} na_n z^n &= \left[-pe^{-i\beta} + \sum_{k=1}^{\infty} C_k z^k \right] \left[\frac{e^{i\beta}}{z^p} + \sum_{n=-(p-1)}^{\infty} b_n z^n \right] \\ &= \frac{-p}{z^p} - pe^{-i\beta} \left[\sum_{n=-(p-1)}^{\infty} b_n z^n \right] + e^{i\beta} \left[\sum_{k=1}^{\infty} C_k z^{k-p} \right] \\ &\quad + \sum_{n=-(p-2)}^{\infty} \left[\sum_{k=1}^{n+p-1} C_k b_{n-k} \right] z^n. \end{aligned}$$

Thus, for $n \geq 1$

$$(3.2) \quad na_n = -pe^{-i\beta}b_n + e^{i\beta}C_{p+n} + \sum_{k=1}^{n+p-1} C_k b_{n-k}.$$

Now

$$C_k = \frac{1}{r^k \pi} \int_0^{2\pi} U(re^{i\theta}) e^{-ik\theta} d\theta.$$

Substituting into (3.2), we obtain

$$\begin{aligned} na_n &= -pe^{-i\beta}b_n + \frac{1}{\pi} \int_0^{2\pi} \left[U(re^{i\theta}) \frac{e^{-in\theta}}{r^n} \right] \\ &\quad \times \left[\frac{e^{i\beta}}{r^p e^{ip\theta}} + \sum_{k=1}^{n+p-1} r^{n-k} e^{i(n-k)\theta} b_{n-k} \right] d\theta \\ &= -pe^{-i\beta}b_n + \frac{1}{\pi} \int_0^{2\pi} \left[U(re^{i\theta}) \frac{e^{-in\theta}}{r^n} \right] \\ &\quad \times \left[f(re^{i\theta}) - \sum_{k=n}^{\infty} b_k (re^{i\theta})^k \right] d\theta. \end{aligned}$$

THEOREM 5. Let

$$F(z) = \sum_{n=-p}^{\infty} a_n z^n \quad (0 < |z| < 1) \quad (a_{-p} \neq 0)$$

be in $\mathcal{K}^*(p)$, then $|a_n| = O(n^{-1})$.

Proof. We may assume without loss of generality that $a_{-p} = 1$. There exists, by Lemma 2,

$$f(z) = \frac{e^{i\beta}}{z^p} + \sum_{n=-(p-1)}^{\infty} b_n z^n \quad (0 < |z| < 1)$$

in $S^*(p)$ such that

$$\left[\operatorname{Re} \frac{zF'(z)}{f(z)} \right] > 0 \quad (|z| < 1).$$

Let $U(z) = \operatorname{Re} [zF'(z)/f(z)]$, then by a well-known result on harmonic functions,

$$\frac{1}{\pi} \int_0^{2\pi} U(re^{i\theta}) d\theta = 2U(0) = -2p \cos \beta \leq 2p.$$

By Lemma 4, we have for $n \geq 1$

$$\begin{aligned} (3.3) \quad n|a_n| &\leq p|b_n| + \frac{1}{\pi r^n} \left| \int_0^{2\pi} U(re^{i\theta}) e^{-in\theta} \sum_{k=n}^{\infty} b_k (re^{i\theta})^k d\theta \right| \\ &\quad + \frac{1}{\pi r^n} \left| \int_0^{2\pi} U(re^{i\theta}) f(re^{i\theta}) e^{-in\theta} d\theta \right| \\ &\leq p|b_n| + \frac{2p}{r^n} \sum_{k=n}^{\infty} |b_k| r^k + \frac{1}{\pi r^n} \int_0^{2\pi} U(re^{i\theta}) |f(re^{i\theta})| d\theta. \end{aligned}$$

The Area Theorems of Golusin [2] and Kobori [3] give for $n \geq 1$

$$\sum_{k=n}^{\infty} k |b_k|^2 \leq \sum_{k=1}^{\infty} k |b_k|^2 \leq \sum_{k=-p}^{-1} |k| |b_k|^2.$$

We thus have,

$$\begin{aligned} \frac{2p}{r^n} \sum_{k=n}^{\infty} |b_k| r^k &\leq \frac{2p}{r^n} \left[\sum_{k=n}^{\infty} k |b_k|^2 \right]^{1/2} \left[\sum_{k=n}^{\infty} r \frac{2k}{k} \right]^{1/2} \\ (3.4) \quad &\leq \frac{2p}{r^n} \left[\sum_{k=-p}^{-1} |k| |b_k|^2 \right]^{1/2} \left[\frac{1}{n} \sum_{k=n}^{\infty} r^{2k} \right]^{1/2} \\ &= 2p \left[\sum_{k=-p}^{-1} |k| |b_k|^2 \right]^{1/2} [n(1-r^2)]^{-1/2}. \end{aligned}$$

Also for $n \geq p$, by Lemma 3

$$\begin{aligned} (3.5) \quad |b_n| &\leq \frac{2p}{(p+n)\sqrt{p}} \left[\sum_{k=-p}^{-1} |k| |b_k|^2 \right]^{1/2} \\ &\leq \frac{1}{\sqrt{p}} \left[\sum_{k=-p}^{-1} |k| |b_k|^2 \right]^{1/2}. \end{aligned}$$

Since $[f(z)]^{-1}$ is p -valently star like we have

$$|f(re^{i\theta})|^{-1} \geq \frac{r^p}{1+r^{2p}}$$

or

$$|f(re^{i\theta})| \leq \frac{(1+r)^{2p}}{r^p}.$$

Therefore, for $n \geq p$

$$\begin{aligned} (3.6) \quad &\frac{1}{\pi r^n} \int_0^{2\pi} U(re^{i\theta}) |f(re^{i\theta})| d\theta \\ &\leq \frac{(1+r)^{2p}}{r^{p+n}} \frac{1}{\pi} \int_0^{2\pi} U(re^{i\theta}) d\theta \\ &\leq \frac{2p(1+r)^{2p}}{r^{p+n}} \leq \frac{2p 4^p}{r^{2n}}. \end{aligned}$$

From (3.3), (3.4), (3.5) and (3.6) we have for $n \geq p$ and any $r < 1$

$$n |a_n| \leq \left[\sqrt{p + 2p [n(1-r^2)]^{-1/2}} \right] \left[\sum_{k=-p}^{-1} |k| |b_k|^2 \right]^{1/2} + 2p 4^p r^{-2n}.$$

Let $r^2 = (1 - 1/n)$, then for $n \geq p + 1$

$$\begin{aligned} n|a_n| &\leq (\sqrt{p} + 2p) \left[\sum_{k=-p}^{-1} |k| |b_k|^2 \right]^{1/2} + 2p 4^p (1 + 1/(n-1))^n \\ &\leq (\sqrt{p} + 2p) \left[\sum_{k=-p}^{-1} |k| |b_k|^2 \right]^{1/2} + 2p 4^p \frac{(p+1)}{p} e. \end{aligned}$$

Thus, $|a_n| = O(n^{-1})$.

REFERENCES

1. M. Cartwright, *Some inequalities in the theory of functions*, Math. Ann. **111** (1935), 98–118.
2. G. M. Golusin, *On p -valent functions*, Mat. Sb. (N. S.) **8** (1940), 277–284.
3. A. Kobori, *Zur Theorie der mehrwertigen Funktionen*, Japan. J. Math. **19** (1947), 301–319.
4. R. J. Libera and M. S. Robertson, *Meromorphic close-to-convex functions*, Michigan Math. J. **8** (1961), 167–175.
5. A. E. Livingston, *P -valent close-to-convex functions*, Trans. Amer. Math. Soc. **115** (1965), 161–179.
6. ———, *p -valent close-to-convex functions*, Thesis, Rutgers, The State University, New Brunswick, N.J., 1963.
7. C. H. Pommerenke, *Über einige klassen meromorpher schlichter Funktionen*, Math. Z. **78**, (1962), 263–284.
8. W. C. Royster, *Meromorphic starlike multivalent functions*, Trans. Amer. Math. Soc. **107** (1963), 300–308.

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